Free Vibration Analysis of Circular Curved Beam by Spectral Element Method

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Free Vibration Analysis of Circular Curved Beam by Spectral Element Method

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Dedicated to

The Almighty God

&

My Father

whose blessings have made

this thesis a reality...

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Certificate

This is to certify that the work in the thesis entitled *Free Vibration Analysis of Circular Curved Beam by Spectral Element Method* by *Pranab Kumar Ojah*, bearing Roll Number 213CE2068, is a record of an original research work carried out by him under my supervision and guidance in partial fulfillment of the requirements for the award of the degree of *Master of Technology* in *Structural Engineering, Department of Civil Engineering*. Neither this thesis nor any part of it has been submitted for any degree or academic award elsewhere.

Manoranjan Barik

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Abstract

The free vibration analysis of a curved beam with various boundary conditions has been studied by many researchers using different computational methods of analysis such as Quadrature Element Method (QEM), Differential Quadrature Element Method (DQEM), Finite Element Method (FEM), Differential Transformation Element Method (DTEM) etc., besides the conventional analytical methods and exact methods. In this work the Spectral Element Method (SEM) for both out-of-plane (transverse) and in-plane free vibration of a curved beam with various boundary conditions is presented considering the effects of rotary inertia and shear deformation. Initially, spectral element matrices are derived for both out-of-plane and in-plane vibration from the governing differential equations of motion in the local polar coordinate system and thereafter transformed to the global Cartesian coordinate system.

Higher accuracy of the natural frequencies with consideration of least degrees of freedom is possible using SEM, thus proving the method to be of very high computational efficiency. To confirm the validity of this method some numerical examples are presented and the results are compared with the other existing solutions.

***Keywords*:** In-plane vibration; Out-of-plane vibration; Spectral Element Method (SEM); Timoshenko curved beam

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Chapter 1

Inroduction

The beam which is originally curved in a plan before applying any load is termed as a curved beam. There are wide applications of the curved beam in various fields of engineering such as civil, mechanical, aerospace engineering. For example in civil engineering, curved beams are used in arch structures, roof structures and circumferential stiffening girder for shells. In Mechanical engineering they are often used in c-clampers, crane hooks, frames of presses, chains, links and rings, spring design, turbo machinery blades in piping system of chemical plants, brake shoes in drum brakes. In Aerospace Engineering they have different applications such as propeller of a helicopter, wings of an airplane and curved wires in missile guidance floated gyroscope.

Vibration is a mechanical phenomenon due to which oscillation of an object occurs about the equilibrium position. It is the dynamic characteristics of a structure. In the field of engineering, prediction of dynamic characteristics of a structure are of great importance. At the present time the aim of the researchers are towards the achievement of more accurate results in an efficient and economic manner. In this context the Spectral Element Method (SEM) has been found to be very promising.

1.1 Spectral Element Method

The history of spectral analysis (also known as Fourier analysis or frequency domain analysis) began with the pioneer work *"Theorie analytique de la chaleur (The analytical theory of heat)"* by Joseph Fourier published in 1822. Later spectral analysis followed continuous development by the contribution of various mathematicians. The fundamental concepts of the Spectral Element Method (SEM) was first introduced by Narayanan and Beskos [1] in 1978. They derived an exact dynamic stiffness matrix for the beam element and used fast Fourier transform (FFT) for the dynamic analysis of plane frame-works. Later Doyle [2] used the terminology "Spectral Element Method" for the first time for the DFT/FFT-based spectral element analysis approach.

The Spectral Element Method (SEM) is a combination of the conventional Finite Element Method (FEM), Spectral Analysis Method (SAM) and Dynamic Stiffness Method (DSM). Assembling of the matrices and meshing are featured from FEM. Exactness of the dynamic stiffness matrix formulated with minimum degrees of freedom is taken from DSM. Superposition of wave modes via DFT and FFT algorithm is included from SAM. In SEM, the exact dynamic stiffness matrices are considered as the element stiffness matrices for the finite number of elements of a structure. In classical DSM for the formulation of the exact dynamic stiffness matrix, the dynamic responses of a structure are usually considered as the harmonic solution of a single frequency. However, in SEM, the dynamic responses are considered as the superposition of a finite number of wave modes of different discrete frequencies based on the DFT theory. Therefore, the computation of the exact dynamic stiffness matrix must be repeated at all discrete frequencies up to the highest frequency of interest. Similar to the conventional FEM, SEM is also an element method. Thus meshing can also be applied in SEM as per the requirements, such as for geometric or material discontinuities of the structure, for the existence of any externally applied forces etc.

1.2 Objectives

The primary objectives of this research work are summarized as follows :

1. To develop circular curved beam element considering the effects of shear deformation and rotary inertia.

2. To develop dynamic stiffness matrix of the circular curved beam element by using Spectral Element Method (SEM).

3. To obtain free out-of-plane and in-plane vibrations of circular curved beam of different shapes and cross sections with various boundary conditions.

Chapter 2

Literature Review

2.1 Out-of-Plane Free Vibration of Curved Beam

So far, many researchers have worked in the field of the out-of-plane vibration of curved beam. Some of those works in context to the present work are briefly summerised in the following.

At the earlier time Love [3] and Ojalvo [4] used classical beam theory to obtain the natural frequencies of vibration of curved beam with the effects of shear deformation and/or rotary inertias neglected. Rao [5] developed the governing differential equations for the coupled bending and torsional vibrations of a curved beam from the Hamilton's principle and solved for the natural frequencies of circular rings and arcs. Davis et al. [6] presented the formulation of curved beam element of constant curvature based on the exact differential equations of an infinitesimal element in static equilibrium. The stiffness and mass matrices were developed from the force-displacement relations and the kinetic energy equations respectively taking the effect of shear deformation and rotary inertia. The matrices were derived based on the local Cartesian coordinate system and then transformed into global coordinate system before the assembling of element matrices. Irie et al. [7] presented the transfer matrix method to study the out-of-plane free vibration of uniform circular arcs based on Timoshenko beam theory. They numerically calculated the natural frequencies (dimensionless) for curved beam with different combination of boundary conditions. Issa [8] analytically derived the dynamic stiffness matrix of a circular curved member of constant section and determined the natural frequencies of continuous Timoshenko curved beam on Winkler type foundations. Irie's equations of motion were used by Kang et al. [9] and they computed the fundamental natural frequencies of out-of-plane vibration by differential quadrature method. This method provided high accuracy with less number of grids. Howson and Jemah [10] developed exact dynamic stiffness matrix from the governing differential equations of motion and formulated a transcendental eigenvalue problem. Values obtained from this method were considered to be the exact values of natural frequencies of out-of-plane vibration. To study the out-of-plane vibration of curved beams with non-uniform cross-section Lee and Chao [11] used an exact method. They derived the governing differential equations for the curved beam of constant radius introducing some physical parameters for simplifying the analysis. Hongjing et al. [12] used differential quadrature element method (DQEM) for the calculation of out-of-plane natural frequencies of continuous horizontally curved girder bridges. The calculated frequencies of the girder were compared with the existing exact solution.The effects of transverse shear deformation, transverse rotary inertia and torsional rotary inertia were included in the study of thin circular bem element by Kim et al. [13]. The stiffness and mass matrices were derived respectively from the strain energy and kinetic energy by using the natural shape function. The matrices developed were based on the local polar coordinate system which were transformed into global Cartesian coordinate system before assembling. Rajasekaran [14] derived the shape functions for nodal variables of a curved beam with non-uniform cross-section by differential transformation method. He studied the static and free vibration of axially functionally graded tapered curved beam considering the effects of shear deformation and rotary inertia.

2.2 In-Plane Free Vibration of Curved Beam

Love [3] and Hartog [15] derived the fundamental governing equations of in-plane vibration of curved beam. Later, Volterra and Morell [16] used the Rayleigh-Ritz method to study the in plane vibration of arcs. The analytical method was used by Takahashi [17] to study the free in-plane vibration of curved beam with both ends clamped. Sabir and Ashwell [18] and Dawe [19] studied the curved beams for in-plane vibration by using finite element method. A general dynamic three-moment equation from the classical equations of motion for the continuous curved beam was derived by Chen [20] and the natural frequencies for periodically-supported rings and continuous curved beam with different boundary conditions were determined. However, all the works mentioned above were based on classical beam theory where shear deformation and rotary inertia were not taken into consideration, and therefore they provided inaccurate results for the vibration characteristics specifically for the higher modes. Taking into account both of the shear deformation and rotary inertia effects so many researchers studied the free in-plane vibration of curved beam. Rao and Sundararajan [21] developed a governing equation of motion for free in-plane vibration of a circular ring considering the effects of shear deformation and rotary inertia. Davis et al. [6] derived the stiffness and mass matrices respectively from the force-displacement relation and kinetic energy equations for the in-plane vibration of a thick curved beam with the effects of shear deformation, rotary inertia and a thin curved beam without those effects. Irie et al.([22], [23]) presented the transfer matrix method to study the in-plane free vibration of uniform and non-uniform circular arcs based upon Timoshenko beam theory. They numerically calculated the natural frequencies (dimensionless) for curved beam with different combination of boundary conditions. Iries equations of motion were extended by Kang et al. [9] and they computed fundamental frequency parameter by differential quadrature method. Lee [24] presented the in-plane free vibration of circular curved Timoshenko beam by pseudospectral method. The analysis was based on Chebyshev polynomials and he used basis function for the boundary conditions. A thin circular beam finite element considering the effects of shear deformation and rotary inertia was developed by Kim et al. [25] to determine the natural frequencies. They derived the stiffness and mass matrices from the strain energy and kinetic energy respectively by using the natural shape function. The matrices developed were based on the local polar coordinate system which were transformed into global Cartesian coordinate system for assembling. Yang et al. [26] studied the free in-plane vibration of uniform and non-uniform curved beam with variable curvature considering the effects of shear deformation, rotary inertia and axis extensibility. They derived the differential equations using the extended-Hamilton principle and solved numerically using Galerkin finite element method.

Chapter 3

Spectral Element model for Circular Curved Beam

3.1 Formulation of the Dynamic Stiffness Matrix

The spectral element formulation for out-of-plane and in-plane free vibration of circular curved beam element begins with the governing differential equations of motion. The formulation is fairly similar to the conventional finite element formulation. The major difference in spectral element formulation is that the governing differential equation of motion is transformation from time domain to frequency domain by using Discrete Fourier Transform (DFT). The time variable disappears and the frequency becomes a parameter to transform the original time domain partial differential equations into the frequency-domain ordinary differential equations. The frequency-domain ordinary differential equations are then solved exactly and the exact wave solutions are used to derive frequency-dependent dynamic shape functions. The exact dynamic stiffness matrix called the spectral element matrix is finally formulated by using the dynamic shape functions [27]. The spectral element matrices are then transformed and assembled to form the global dynamic stiffness matrix where the end restrains are applied similar to the conventional finite element method thus producing the reduced dynamic stiffness matrix. The determinant of the reduced dynamic stiffness matrix which is a function of the natural frequencies when equated to zero and solved gives rise to the required natural frequencies.

3.1.1 Spectral Element Matrix for Out-of-Plane Free Vibration of Curved Beam

The out-of-plane free vibration of a horizontally circular curved beam element neglecting damping and warping with radius 'R' is shown in the Fig. 3.1; where v, ϕ, ψ are the transverse deflection, angle of rotation and angle of torsion respectively.

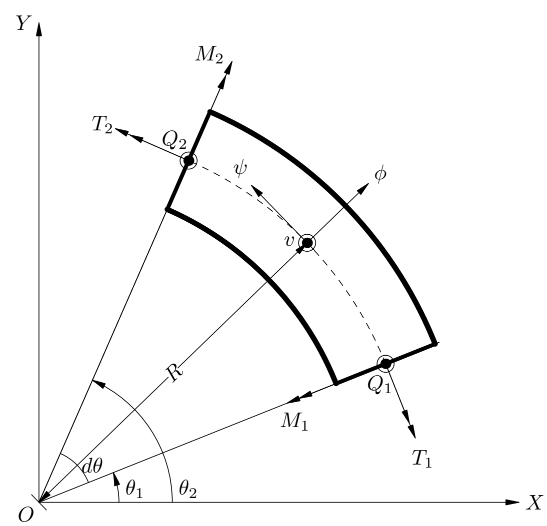

Figure 3.1: Curved beam element for transverse vibration

The governing differential equations of motion for the out-of-plane vibration of a circular curved beam based on the Timoshenko beam theory are written as [7] :

$$\frac{1}{R^2}\frac{dQ_y^*}{d\theta} + \rho A\omega^2 v = 0 \tag{3.1}$$

$$-\frac{1}{R}\frac{dM_x^*}{d\theta} + \frac{T^*}{R} - Q_y^* + \rho I_x\omega^2\phi = 0 \tag{3.2}$$

$$\frac{1}{R}\frac{dT^*}{d\theta} + \frac{M_x^*}{R} + \rho J_z \omega^2 \psi = 0 \tag{3.3}$$

where M_x^* is the bending moment, T^* is the torsional moment, Q_y^* is the shear force, A is the cross-sectional area, I_x is the moment of inertia of the cross-section, J_z is the polar moment of inertia and ρ is the density of the material.

From the elementary beam theory the bending moment, torsional moment and shear force are given as :

$$M_x^* = \frac{EI_x}{R}\left(-\psi - \frac{d\phi}{d\theta}\right) \tag{3.4}$$

$$Q_y^* = \kappa AG\left(\phi + \frac{dv}{d\theta}\right) \tag{3.5}$$

$$T^* = \frac{GC_z}{R}\left(-\phi + \frac{d\psi}{d\theta}\right) \tag{3.6}$$

where E, G, C_z and κ are the Young's modulus, shear modulus, torsional moment of inertia [28] and shear coefficient of the cross-section [29] respectively dependant on the material property and shape of the cross-section.

Substituting the Eqs. (3.4),(3.5),(3.6) into Eqs. (3.1),(3.2),(3.3) yields :

$$\kappa\frac{G}{E}s_x^2 v'' + \lambda^2 v + \kappa\frac{G}{E}s_x^2\phi' = 0 \tag{3.7}$$

$$-\kappa\frac{G}{E}s_x^2 v' + (1+\mu)\psi' + \phi'' - \left(\mu + \kappa\frac{G}{E}s_x^2 - \lambda^2\frac{1}{s_x^2}\right)\phi = 0 \tag{3.8}$$

$$\mu\psi'' - \left\{1 - \lambda^2\left(\frac{1}{s_x^2} + \frac{1}{s_y^2}\right)\right\}\psi - (1+\mu)\phi' = 0 \tag{3.9}$$

For simplicity, the following dimensionless variables are introduced :

$$\begin{aligned}(M_x, T) &= \frac{R}{EI_x}(M_x^*, T^*), \qquad & Q_y &= \frac{R^2}{EI_x}Q_y^*,\\ \lambda^2 &= \frac{\rho A R^4 \omega^2}{EI_x}, \qquad & \mu &= \frac{GC_z}{EI_x},\\ s_x^2 &= \frac{AR^2}{I_x}, \qquad & s_y^2 &= \frac{AR^2}{I_y}\end{aligned}$$

where s_x and s_y are the slenderness ratios, μ is the rigidity ratio and λ denotes a frequency parameter.

Using the above dimensionless parameters we can re-write the expressions for bending moment, torsional moment and shear force as :

$$M_x = -\psi - \frac{d\phi}{d\theta} \tag{3.10}$$

$$Q_y = \kappa \frac{G}{E} s_x^2 \left(\phi + \frac{dv}{d\theta} \right) \tag{3.11}$$

$$T = \mu \left(-\phi + \frac{d\psi}{d\theta} \right) \tag{3.12}$$

In the matrix form we can write the Eqs. (3.10),(3.11) and (3.12) as :

$$\mathbf{F} = \mathbf{C}\mathbf{U} \tag{3.13}$$

$$\text{where} \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 & 0 \\ \kappa \frac{G}{E} s_x^2 & 0 & 0 & 0 & \kappa \frac{G}{E} s_x^2 & 0 \\ -\mu & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{3.14}$$

$\mathbf{U} = \{\phi \quad v \quad \psi \quad \phi' \quad v' \quad \psi'\}^T, \qquad \mathbf{F} = \{M \quad Q \quad T\}^T$

Assuming the general solutions of Eqs.(3.7), (3.8) and (3.9) to be :

$$\phi(\theta) = a e^{-ip\theta}, \qquad v(\theta) = \alpha a e^{-ip\theta}, \qquad \psi(\theta) = \beta a e^{-ip\theta} \tag{3.15}$$

where p is the wavenumber. Substitution of Eq.(3.15) into Eqs.(3.7), (3.8) and (3.9) gives an eigenvalue problem as:

$$\mathbf{X}(\mathbf{p}) \begin{Bmatrix} 1 \\ \alpha \\ \beta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.16}$$

where,

$$\mathbf{X}(\mathbf{p}) = \begin{bmatrix} -ip\kappa \frac{G}{E} s_x^2 & -p^2 \kappa \frac{G}{E} s_x^2 + \lambda^2 & 0 \\ -p^2 - \mu - \kappa \frac{G}{E} s_x^2 + \frac{\lambda^2}{s_x^2} & ip\kappa \frac{G}{E} s_x^2 & -ip(1+u) \\ ip(1+u) & 0 & -p^2\mu - 1 + \lambda^2 \left(\frac{1}{s_x^2} + \frac{1}{s_y^2} \right) \end{bmatrix} \tag{3.17}$$

From Eq.(3.16), we get a dispersion relation in the form of:

$$c_1 p^6 + c_2 p^4 + c_3 p^2 + c_4 = 0 \tag{3.18}$$

Solving Eq. (3.18) we get six wavenumbers $p_1, p_2 \cdots p_6$.

From Eq.(3.16) α_i and β_i for each wavenumber $p_i (i = 1, 2, \cdots, 6)$ are determined. By using the six wavenumbers, we can write the general solution as :

$$\begin{aligned} \phi(\theta) &= \sum_{i=1}^{6} a_i e^{-ip_i\theta} = \mathbf{e}_\phi(\theta, \lambda)\mathbf{a} \\ v(\theta) &= \sum_{i=1}^{6} \alpha_i a_i e^{-ip_i\theta} = \mathbf{e}_\mathbf{v}(\theta, \lambda)\mathbf{a} \\ \psi(\theta) &= \sum_{i=1}^{6} \beta_i a_i e^{-ip_i\theta} = \mathbf{e}_\psi(\theta, \lambda)\mathbf{a} \end{aligned} \tag{3.19}$$

where $\mathbf{a} = \{a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6\}^T$ and

$$\begin{aligned} \mathbf{e}_\phi(\theta, \lambda) &= [\ e^{-ip_1\theta} \quad e^{-ip_2\theta} \quad e^{-ip_3\theta} \quad e^{-ip_4\theta} \quad e^{-ip_5\theta} \quad e^{-ip_6\theta}\ ] \\ \mathbf{e}_\mathbf{v}(\theta, \lambda) &= \mathbf{e}_\phi(\theta, \lambda)\mathbf{A}(\lambda) \\ \mathbf{e}_\psi(\theta, \lambda) &= \mathbf{e}_\phi(\theta, \lambda)\mathbf{B}(\lambda) \end{aligned} \tag{3.20}$$

with

$$\begin{aligned} \mathbf{A}(\lambda) &= diag[\alpha_j(\lambda)], \qquad \alpha_j(\lambda) = \alpha[p_j(\lambda)] \\ \mathbf{B}(\lambda) &= diag[\beta_j(\lambda)], \qquad \beta_j(\lambda) = \beta[p_j(\lambda)] \end{aligned} \tag{3.21}$$

where $j = (1; 2; \cdots; 6)$.

The degrees of freedoms and the forces at the two ends may be written as :

$$\mathbf{d}(\lambda) = \begin{Bmatrix} \phi_1 \\ v_1 \\ \psi_1 \\ \phi_2 \\ v_2 \\ \psi_2 \end{Bmatrix} = \begin{Bmatrix} \phi(0) \\ v(0) \\ \psi(0) \\ \phi(\Theta) \\ v(\Theta) \\ \psi(\Theta) \end{Bmatrix} \tag{3.22}$$

$$\mathbf{f_c}(\lambda) = \begin{Bmatrix} M_1 \\ Q_1 \\ T_1 \\ M_2 \\ Q_2 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -M(0) \\ -Q(0) \\ -T(0) \\ +M(\Theta) \\ +Q(\Theta) \\ +T(\Theta) \end{Bmatrix} \tag{3.23}$$

Substitution of Eq. (3.19) into Eq. (3.22) yields the relationship :

$$\mathbf{d} = \mathbf{H}(\lambda)\mathbf{a} \tag{3.24}$$

where

$$\mathbf{H}(\lambda) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \alpha_1 e_1 & \alpha_2 e_2 & \alpha_3 e_3 & \alpha_4 e_4 & \alpha_5 e_5 & \alpha_6 e_6 \\ \beta_1 e_1 & \beta_2 e_2 & \beta_3 e_3 & \beta_4 e_4 & \beta_5 e_5 & \beta_6 e_6 \end{bmatrix} \tag{3.25}$$

with the use of the definition $e_j = e^{-ip_j\theta}, \quad (j = 1, 2, \cdots, 6)$

By using Eq. (3.24), the coefficient vector $\mathbf{a}$ can be eliminated from Eq.(3.19) to obtain :

$$\begin{aligned} \phi(\theta) &= \mathbf{N}_\phi(\theta;\lambda)\mathbf{d} \\ v(\theta) &= \mathbf{N_v}(\theta;\lambda)\mathbf{d} \\ \psi(\theta) &= \mathbf{N}_\psi(\theta;\lambda)\mathbf{d} \end{aligned} \tag{3.26}$$

where $\mathbf{N}_\phi(\theta;\lambda)$, $\mathbf{N_v}(\theta;\lambda)$, and $\mathbf{N}_\psi(\theta;\lambda)$ are the dynamic shape functions defined by

$$\begin{aligned} \mathbf{N}_\phi(\theta;\lambda) &= \mathbf{e}_\phi(\theta,\lambda)\mathbf{H^{-1}}(\lambda) \\ \mathbf{N_v}(\theta;\lambda) &= \mathbf{e}_\phi(\theta,\lambda)\mathbf{A}(\lambda)\mathbf{H^{-1}}(\lambda) \\ \mathbf{N}_\psi(\theta;\lambda) &= \mathbf{e}_\phi(\theta,\lambda)\mathbf{B}(\lambda)\mathbf{H^{-1}}(\lambda) \end{aligned} \tag{3.27}$$

Substitution of Eq. (3.26) into Eq.(3.13) and the results into Eq.(3.23) gives the relationship between $\mathbf{f_c}$ and $\mathbf{d}$ as :

$$\mathbf{S}(\lambda)\mathbf{d} = \mathbf{f_c}(\lambda) \tag{3.28}$$

where $\mathbf{S}(\lambda)$ is the spectral element (dynamic stiffness) matrix and is given by :

$$\mathbf{S}(\lambda) = \begin{bmatrix} -\mathbf{C} & 0 \\ 0 & \mathbf{C} \end{bmatrix} \mathbf{R}(\lambda)\, \mathbf{H^{-1}}(\lambda) \tag{3.29}$$

where $\mathbf{C}, \mathbf{H}(\lambda)$ are taken from Eq.(3.14) and (3.25) respectively.

$$\mathbf{R}(\lambda) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \gamma_1\alpha_1 & \gamma_2\alpha_2 & \gamma_3\alpha_3 & \gamma_4\alpha_4 & \gamma_5\alpha_5 & \gamma_6\alpha_6 \\ \gamma_1\beta_1 & \gamma_2\beta_2 & \gamma_3\beta_3 & \gamma_4\beta_4 & \gamma_5\beta_5 & \gamma_6\beta_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \alpha_1 e_1 & \alpha_2 e_2 & \alpha_3 e_3 & \alpha_4 e_4 & \alpha_5 e_5 & \alpha_6 e_6 \\ \beta_1 e_1 & \beta_2 e_2 & \beta_3 e_3 & \beta_4 e_4 & \beta_5 e_5 & \beta_6 e_6 \\ \gamma_1 e_1 & \gamma_2 e_2 & \gamma_3 e_3 & \gamma_4 e_4 & \gamma_5 e_5 & \gamma_6 e_6 \\ \gamma_1\alpha_1 e_1 & \gamma_2\alpha_2 e_2 & \gamma_3\alpha_3 e_3 & \gamma_4\alpha_4 e_4 & \gamma_5\alpha_5 e_5 & \gamma_6\alpha_6 e_6 \\ \gamma_1\beta_1 e_1 & \gamma_2\beta_2 e_2 & \gamma_3\beta_3 e_3 & \gamma_4\beta_4 e_4 & \gamma_5\beta_5 e_5 & \gamma_6\beta_6 e_6 \end{bmatrix} \tag{3.30}$$

where $\gamma_j = -ip_j, \quad (j = 1, 2, \cdots 6)$.

3.1.2 Spectral Element Matrix for In Plane Free Vibration of Curved Beam

The in-plane free vibration of a horizontally circular curved beam element of uniform cross-section with radius ‘R’ is shown in the Fig. 3.2; where u, w and

ψ are the radial displacement, tangential displacement and slope due to pure bending respectively.

The governing differential equations of motion for the in-plane vibration of a circular curved beam based on the Timoshenko beam theory are written as [22] :

$$\frac{1}{R}\frac{dQ_x^*}{d\theta} + \frac{1}{R}N^* + \rho AR\omega^2(1+k^2)u = 0 \tag{3.31}$$

$$\frac{1}{R^2}\frac{dM_y^*}{d\theta} + \frac{1}{R}Q_x^* + \rho AR\omega^2(k_1^2 w + k_2^2\psi) = 0 \tag{3.32}$$

$$\frac{1}{R}\frac{dN^*}{d\theta} - \frac{1}{R}Q_x^* + \rho AR\omega^2\{(1+k^2)w + k_1^2\psi\} = 0 \tag{3.33}$$

where N^*, Q_x^*, M_y^* are the tensile force, the shear force and the bending moment respectively. Also A is the cross-sectional area, I_x is the moment of inertia of the cross-section and ρ is the density of the material.

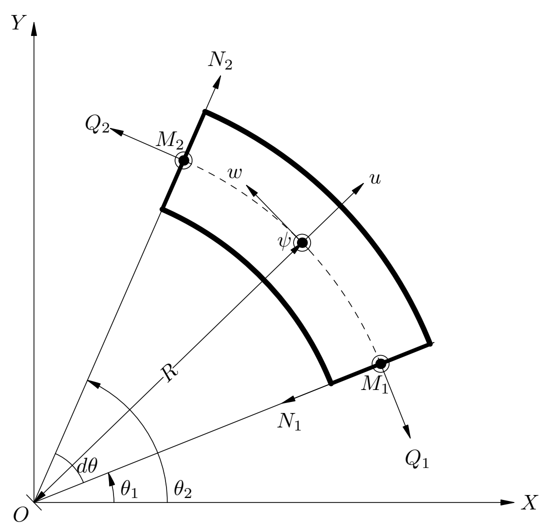

Figure 3.2: Curved beam element for in-plane vibration

The quantities k^2, k_1^2 and k_2^2 are the dimensionless parameters defined as :

$$k^2 = (d/4R)^2, \qquad k_1^2 = k^2(1+k^2), \qquad k_2^2 = k^2(1+4k^2+k^4)$$

for a curved beam with circular cross-section of diameter d, radius R and

$$k^2 = (h/2R)\coth(h/2R) - 1, \qquad k_1^2 = k^2(1+k^2) + (1/3)(h/2R)^2,$$
$$k_2^2 = k^2[k^2+k^4+(h/2R)^2] + (1/3)(h/2R)^2$$

for a curved beam with rectangular cross-section of height h, radius R.

From the elementary theory of beam the tensile force, shear force and bending moment are given as :

$$N^* = EA\left(\frac{dw}{d\theta} - u\right) \tag{3.34}$$

$$Q_x^* = \kappa GA\left(w + \frac{du}{d\theta} - \psi\right) \tag{3.35}$$

$$M_y^* = EARk^2\frac{d\psi}{d\theta} \tag{3.36}$$

where E, G, κ are the Young's modulus, shear modulus and the shear coefficient of the cross-section [29] respectively dependent on the material property and the shape of the cross section.

Substituting the Eqs. (3.34),(3.35),(3.36) into Eqs. (3.31),(3.32),(3.33) yields :

$$\frac{\kappa}{2(1+\nu)}u'' - \left(1 - \lambda^2\frac{1+k^2}{s_y^2}\right)u - \frac{\kappa}{2(1+\nu)}\psi' + \left\{1 + \frac{\kappa}{2(1+\nu)}\right\}w' = 0 \tag{3.37}$$

$$\frac{\kappa}{2(1+\nu)}u' + k^2\psi'' - \left\{\frac{\kappa}{2(1+\nu)} - \lambda^2\frac{k_2^2}{s_y^2}\right\}\psi + \left\{\frac{\kappa}{2(1+\nu)} + \lambda^2\frac{k_1^2}{s_y^2}\right\}w = 0 \tag{3.38}$$

$$\left\{1 + \frac{\kappa}{2(1+\nu)}\right\}u' - \left\{\frac{\kappa}{2(1+\nu)} + \lambda^2\frac{k_1^2}{s_y^2}\right\}\psi - w'' + \left\{\frac{\kappa}{2(1+\nu)} - \lambda^2\frac{1+k^2}{s_y^2}\right\}w = 0 \tag{3.39}$$

Introducing the following dimensionless variables:

$$(N, Q_x, M_y) = \frac{1}{EA}(N^*, Q_x^*, \frac{1}{R}M_y^*),$$
$$s_y^2 = \frac{AR^2}{I_y}, \qquad \lambda^2 = \frac{\rho AR^4\omega^2}{EI_y}$$

where s_y is the slenderness ratio and λ is the frequency parameter.

Using the above dimensionless parameters we can rewrite the expressions for tensile force, shear force and bending moment as :

$$N = \frac{dw}{d\theta} - u \tag{3.40}$$

$$Q_x = \frac{\kappa}{2(1+\nu)}\left(w + \frac{du}{d\theta} - \psi\right) \tag{3.41}$$

$$M_y = k^2 \frac{d\psi}{d\theta} \tag{3.42}$$

In the matrix form we can write the Eqs. (3.40), (3.41) and (3.42) as :

$$\mathbf{F} = \mathbf{C}\mathbf{U} \tag{3.43}$$

where

$$\mathbf{C} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{\kappa}{2(1+\nu)} & -\frac{\kappa}{2(1+\nu)} & \frac{\kappa}{2(1+\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k^2 \end{bmatrix} \tag{3.44}$$

$\mathbf{U} = \{u \quad w \quad \psi \quad u' \quad w' \quad \psi'\}^T, \qquad \mathbf{F} = \{N \quad Q_x \quad M_y\}^T$

Assuming the general solutions of Eqs.(3.37), (3.38) and (3.39) to be :

$$u(\theta) = ae^{-ip\theta}, \qquad w(\theta) = \alpha a e^{-ip\theta}, \qquad \psi(\theta) = \beta a e^{-ip\theta} \tag{3.45}$$

where p is the wavenumber.

Substitution of Eq.(3.45) into Eqs.(3.37), (3.38) and (3.39) gives an eigenvalue problem as:

$$\mathbf{X(p)} \begin{Bmatrix} 1 \\ \alpha \\ \beta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.46}$$

where,

$$\mathbf{X(p)} = \begin{bmatrix} -ip\left\{1 + \frac{\kappa}{2(1+\nu)}\right\} & p^2 + \frac{\kappa}{2(1+\nu)} - \lambda^2 \frac{1+k^2}{s_y^2} & -\left\{\frac{\kappa}{2(1+\nu)} + \lambda^2 \frac{k_1^2}{s_y^2}\right\} \\ -\frac{\kappa}{2(1+\nu)} p^2 - \left(1 - \lambda^2 \frac{1+k^2}{s_y^2}\right) & -ip\frac{\kappa}{2(1+\nu)} & ip\frac{\kappa}{2(1+\nu)} \\ -ip\frac{\kappa}{2(1+\nu)} & \frac{\kappa}{2(1+\nu)} + \lambda^2 \frac{k_1^2}{s_y^2} & -k^2p^2 - \left\{\frac{\kappa}{2(1+\nu)} - \lambda^2 \frac{k_2^2}{s_y^2}\right\} \end{bmatrix} \tag{3.47}$$

From Eq.(3.46), we get a dispersion relation in the form of:

$$c_1 p^6 + c_2 p^4 + c_3 p^2 + c_4 = 0 \tag{3.48}$$

Solving Eq. (3.48) we get six wavenumbers $p_1, p_2 \cdots p_6$.

From Eq.(3.46) α_i and β_i for each wavenumber $p_i (i = 1, 2, \cdots, 6)$ are determined.

By using the six wavenumbers, we can write the general solution as :

$$\begin{aligned} u(\theta) &= \sum_{i=1}^{6} a_i e^{-ip_i\theta} = \mathbf{e_u}(\theta,\lambda)\mathbf{a} \\ w(\theta) &= \sum_{i=1}^{6} \alpha_i a_i e^{-ip_i\theta} = \mathbf{e_w}(\theta,\lambda)\mathbf{a} \\ \psi(\theta) &= \sum_{i=1}^{6} \beta_i a_i e^{-ip_i\theta} = \mathbf{e}_\psi(\theta,\lambda)\mathbf{a} \end{aligned} \tag{3.49}$$

where $\mathbf{a} = \{a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6\}^T$ and

$$\begin{aligned} &\mathbf{e_u}(\theta,\lambda) = [\; e^{-ip_1\theta} \quad e^{-ip_2\theta} \quad e^{-ip_3\theta} \quad e^{-ip_4\theta} \quad e^{-ip_5\theta} \quad e^{-ip_6\theta} \;] \\ &\mathbf{e_w}(\theta,\lambda) = \mathbf{e_u}(\theta,\lambda)\mathbf{A}(\lambda) \\ &\mathbf{e}_\psi(\theta,\lambda) = \mathbf{e_u}(\theta,\lambda)\mathbf{B}(\lambda) \end{aligned} \tag{3.50}$$

with

$$\begin{aligned} \mathbf{A}(\lambda) &= diag[\alpha_j(\lambda)], \qquad \alpha_j(\lambda) = \alpha[k_j(\lambda)] \\ \mathbf{B}(\lambda) &= diag[\beta_j(\lambda)], \qquad \beta_j(\lambda) = \beta[k_j(\lambda)] \end{aligned} \tag{3.51}$$

where $j = (1; 2; \cdots; 6)$.

The degrees of freedom and the forces at the two ends may be written as :

$$\mathbf{d}(\lambda) = \begin{Bmatrix} u_1 \\ w_1 \\ \psi_1 \\ u_2 \\ w_2 \\ \psi_2 \end{Bmatrix} = \begin{Bmatrix} u(0) \\ w(0) \\ \psi(0) \\ u(\Theta) \\ w(\Theta) \\ \psi(\Theta) \end{Bmatrix} \tag{3.52}$$

$$\mathbf{f_c}(\lambda) = \begin{Bmatrix} N_1 \\ Q_1 \\ M_1 \\ N_2 \\ Q_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -N(0) \\ -Q(0) \\ -M(0) \\ +N(\Theta) \\ +Q(\Theta) \\ +M(\Theta) \end{Bmatrix} \tag{3.53}$$

Substitution of Eq. (3.49) into Eq. (3.52) yields the relationship :

$$\mathbf{d} = \mathbf{H}(\lambda)\mathbf{a} \tag{3.54}$$

where

$$\mathbf{H}(\lambda) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \alpha_1 e_1 & \alpha_2 e_2 & \alpha_3 e_3 & \alpha_4 e_4 & \alpha_5 e_5 & \alpha_6 e_6 \\ \beta_1 e_1 & \beta_2 e_2 & \beta_3 e_3 & \beta_4 e_4 & \beta_5 e_5 & \beta_6 e_6 \end{bmatrix} \tag{3.55}$$

with the use of the definition $e_j = e^{-ip_j\theta}, \quad (j = 1, 2, \cdots, 6)$.

By using Eq. (3.54), the coefficient vector $\mathbf{a}$ can be eliminated from Eq.(3.49) to obtain :

$$\begin{aligned} u(\theta) &= \mathbf{N_u}(\theta;\lambda)\mathbf{d} \\ w(\theta) &= \mathbf{N_w}(\theta;\lambda)\mathbf{d} \\ \psi(\theta) &= \mathbf{N}_\psi(\theta;\lambda)\mathbf{d} \end{aligned} \tag{3.56}$$

where $\mathbf{N_u}(\theta;\lambda)$, $\mathbf{N_w}(\theta;\lambda)$, $\mathbf{N}_\psi(\theta;\lambda)$ are the dynamic shape functions and are defined by :

$$\begin{aligned} \mathbf{N_u}(\theta;\lambda) &= \mathbf{e_u}(\theta,\lambda)\mathbf{H^{-1}}(\lambda) \\ \mathbf{N_w}(\theta;\lambda) &= \mathbf{e_u}(\theta,\lambda)\mathbf{A}(\lambda)\mathbf{H^{-1}}(\lambda) \\ \mathbf{N}_\psi(\theta;\lambda) &= \mathbf{e_u}(\theta,\lambda)\mathbf{B}(\lambda)\mathbf{H^{-1}}(\lambda) \end{aligned} \tag{3.57}$$

Substitution of Eq. (3.56) into Eq.(3.43) and the results into Eq.(3.53) gives the relationship between $\mathbf{f_c}$ and $\mathbf{d}$ as :

$$\overline{\mathbf{S}}(\lambda)\mathbf{d} = \mathbf{f_c}(\lambda) \tag{3.58}$$

where $\overline{\mathbf{S}}(\lambda)$ is the spectral element (dynamic stiffness) matrix in local coordinate system and is given by :

$$\overline{\mathbf{S}}(\lambda) = \begin{bmatrix} -\mathbf{C} & 0 \\ 0 & \mathbf{C} \end{bmatrix} \mathbf{R}(\lambda)\mathbf{H^{-1}}(\lambda) \tag{3.59}$$

where $\mathbf{C}, \mathbf{H}(\lambda)$ are taken from Eq.(3.44) and (3.55) respectively.

$$\mathbf{R}(\lambda) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \gamma_1\alpha_1 & \gamma_2\alpha_2 & \gamma_3\alpha_3 & \gamma_4\alpha_4 & \gamma_5\alpha_5 & \gamma_6\alpha_6 \\ \gamma_1\beta_1 & \gamma_2\beta_2 & \gamma_3\beta_3 & \gamma_4\beta_4 & \gamma_5\beta_5 & \gamma_6\beta_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \alpha_1 e_1 & \alpha_2 e_2 & \alpha_3 e_3 & \alpha_4 e_4 & \alpha_5 e_5 & \alpha_6 e_6 \\ \beta_1 e_1 & \beta_2 e_2 & \beta_3 e_3 & \beta_4 e_4 & \beta_5 e_5 & \beta_6 e_6 \\ \gamma_1 e_1 & \gamma_2 e_2 & \gamma_3 e_3 & \gamma_4 e_4 & \gamma_5 e_5 & \gamma_6 e_6 \\ \gamma_1\alpha_1 e_1 & \gamma_2\alpha_2 e_2 & \gamma_3\alpha_3 e_3 & \gamma_4\alpha_4 e_4 & \gamma_5\alpha_5 e_5 & \gamma_6\alpha_6 e_6 \\ \gamma_1\beta_1 e_1 & \gamma_2\beta_2 e_2 & \gamma_3\beta_3 e_3 & \gamma_4\beta_4 e_4 & \gamma_5\beta_5 e_5 & \gamma_6\beta_6 e_6 \end{bmatrix} \tag{3.60}$$

where $\gamma_j = -ip_j, \quad (j = 1, 2, \cdots 6)$.

3.2 Transformation of Dynamic Stiffness Matrix

The dynamic stiffness matrix (spectral element matrix) for each element of the curved beam is calculated based upon the local polar coordinate system as described in the chapter 3. Before the assembling of the matrices each spectral element matrix $\mathbf{S}(\lambda)$ is transformed into the global Cartesian coordinate system $\overline{\mathbf{S}(\lambda)}$ by the transformation :

$$\overline{\mathbf{S}(\lambda)} = \mathbf{T}^T \, \mathbf{S}(\lambda) \, \mathbf{T} \tag{3.61}$$

Where $\mathbf{T}$ is the transformation matrix.
For the out-of-plan vibration of curved beam

$$\mathbf{T} = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

For the in-plan vibration of curved beam

$$\mathbf{T} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 0 & 0 & \sin\theta_2 & 0 & \cos\theta_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3.3 Globalisation of Dynamic Stiffness Matrix

The transformed dynamic stiffness matrix (spectral element matrix), $\overline{\mathbf{S}(\lambda)}$ of each element is now assembled to form spectral global matrix following the similar convention of the classical finite element method where the inter-element continuity conditions are automatically maintained. The classical boundary conditions are applied by eliminating the corresponding rows and columns of those restrained degrees of freedom thus forming the reduced spectral global matrix $\mathbf{S_g}(\lambda)$ and the eigenvalue problem is formed as :

$$\mathbf{S_g}(\lambda)\ \mathbf{d_g} = 0 \tag{3.62}$$

where $\mathbf{d_g}$ is the global spectral nodal degrees of freedoms vector.

3.4 Eigensolution Procedure

The eigenfrequencies $\lambda_i(i = 1, 2, ..., \infty)$ are determined by equating the determinant of $\mathbf{S}_g(\lambda)$ to zero at $\lambda = \lambda_i$, i.e.,

$$\Big|\mathbf{S_g}(\lambda)\Big| = 0 \tag{3.63}$$

The real part of the i-th eigenfrequency λ_i corresponds to the i-th natural frequency of the system. As the global dynamic stiffness matrix $\mathbf{S_g}(\lambda)$ is the transcendental function of frequency, Eq. (3.62) is the transcendental eigenvalue problem. Thus, the various eigensolvers available for the linear eigenvalue problems are no longer applicable. The only way to compute the eigenfrequencies of Eq. (3.63) is to vary the frequencyvin small steps, calculating $\Big|\mathbf{S_g}(\lambda)\Big|$ at each step, and to seek all frequency values where $\Big|\mathbf{S_g}(\lambda)\Big|$ becomes zero. Here for calculating the eigenfrequencies, we have used the determinant search method.

Chapter 4

Numerical Results and Discussion

4.1 Numerical Examples of out-of-plane vibration of curved beam

4.1.1 Natural frequencies of circular curved beam with different boundary conditions

The natural frequencies of the out-of-plane vibration of circular curved beam with uniform cross-section under different boundary conditions are computed using SEM considering all the effects of transverse shear deformation, transverse rotary inertia and torsional rotary inertia.

In the Table 4.1 out-of-plane frequency parameters of circular curved beam with circular cross-section under the free-free, free-simple, free-clamped, simple-simple and simple-clamped boundary conditions are presented for an opening angle of the beam $\Theta = 120°$, shear correction factor $\kappa = 0.89$ for circular cross-section and Poisson's ratio $\nu = 0.3$. The results obtained by SEM are compared with the analytical ones (Irie et al. [7]) which are well in agreement.

Table 4.1: Out-of-plane frequency parameters of circular curved beam with circular cross-section

Slenderness Ratio $s_x = s_y$	Mode Number	Free-Free		Free-Simple		Free-Clamped		Simple-Simple		Simple-Clamped	
		Present (SEM)	Ref. [7]	Present (SEM)	Ref. [7]	Present (SEM)	Ref. [7]	Present (SEM)	Ref. [7]	Present (SEM)	Ref. [7]
10	1	6.826	6.892	5.980	5.967	0.814	-	5.354	5.369	2.176	2.159
	2	7.758	7.855	7.108	7.019	3.008	3.028	6.292	6.285	6.604	6.488
	3	12.589	12.693	11.894	11.714	6.707	6.690	11.692	11.640	9.180	9.189
	4	17.511	17.480	15.783	15.809	10.610	10.523	13.751	13.433	14.455	14.350
100	1	11.440	11.415	8.877	8.789	0.833	-	6.663	6.561	2.395	2.315
	2	22.848	23.044	19.950	19.795	3.537	3.583	17.127	17.140	9.540	9.582
	3	41.875	41.227	37.341	37.167	11.849	11.872	33.150	33.138	21.654	21.193
	4	56.952	56.769	54.643	54.256	24.983	24.993	51.294	51.353	38.133	38.512

The out-of-plane natural frequencies of circular curved beam with rectangular cross-section with various combination of boundary conditions is shown in Table 4.2. For the curved beam, b is the breadth and h is the depth of the rectangular cross-section. Here we have presented the results for the height to depth ratio 2 and for the opening angle Θ of the beam 60°. The shear correction factor κ for rectangular cross-section is 0.85 and Poisson's ratio ν is 0.3. The slenderness ratio of the cross-section in x direction s_x is 20 and 100 and that of in y direction $s_y = \dfrac{b}{h}\, s_x$.

Table 4.2: Out-of-plane frequency parameters of circular curved with rectangular cross-section

Slenderness Ratio s_x	Mode Number	F-F	F-S	F-C	S-S	S-C
20	1	19.379	14.615	3.212	11.453	10.360
	2	27.197	26.617	12.846	24.401	17.043
	3	45.082	38.462	18.855	33.835	34.377
	4	55.346	54.845	40.269	52.765	43.283
100	1	41.360	38.959	3.284	33.350	12.302
	3	52.978	43.719	17.886	40.181	42.784
	3	107.778	93.193	52.166	79.640	79.246
	4	140.282	138.888	79.289	132.75	93.325

(C-Clamped, F-Free, S-Simple Support)

Table 4.3: Out-of-plane frequency parameters of clamped-clamped circular curved beam

Slenderness Ratio $s_x = s_y$		Mode Number	$\Theta = 60°$		$\Theta = 120°$		$\Theta = 180°$	
			Present (SEM)	Reference [7]	Present (SEM)	Reference [7]	Present (SEM)	Reference [7]
Square cross section	20	1	16.744	16.74	4.283	4.282	1.777	1.776
		2	36.965	36.92	11.691	11.69	4.982	4.982
		3	40.452	40.45	22.060	22.05	10.135	10.13
		4	69.621	69.62	22.384	22.38	16.764	16.76
	100	1	19.402	19.40	4.452	4.451	1.804	1.804
		2	54.030	54.03	12.827	12.83	5.198	5.198
		3	105.649	105.6	25.990	25.99	10.919	10.92
		4	172.774	172.8	43.571	43.57	18.726	18.72
Circular cross section	20	1	16.884	16.88	4.309	4.309	1.791	1.791
		2	39.700	39.70	11.796	11.76	5.032	5.032
		3	40.934	40.90	22.510	22.50	10.232	10.23
		4	70.581	70.51	23.303	23.30	16.917	16.91
	100	1	19.454	19.45	4.473	4.473	1.818	1.818
		2	54.148	54.14	12.892	12.89	5.242	5.242
		3	105.861	105.9	26.081	26.08	10.989	10.99
		4	173.158	173.1	43.684	43.68	18.813	18.81

Table 4.3 shows the frequency parameters of out-of-plane vibration of clamped-clamped circular curved beam of square and circular cross sections. The frequencies are for $\Theta = 60°$, $120°$ and $180°$ with slenderness ratios 20 and 100, the shear correction factor $\kappa = 0.85/.89$ (square/ circular cross-section) and Poisson's ratio $\nu = 0.3$. The natural frequencies by SEM compare well with the solutions by transfer matrix method (Irie et al. [7]).

The variation of the natural frequencies of out-of-plane vibration of circular curved beam with square cross-section with respect to the opening angle Θ for various combination of boundary conditions are shown in Fig. 4.1-4.5 (C-Clamped, F-Free, S-Simple Support). The shear correction factor κ for square cross-section is 0.85 and Poisson's ratio ν is 0.3. The slenderness ratios of the cross-section are taken as 20 and 100.

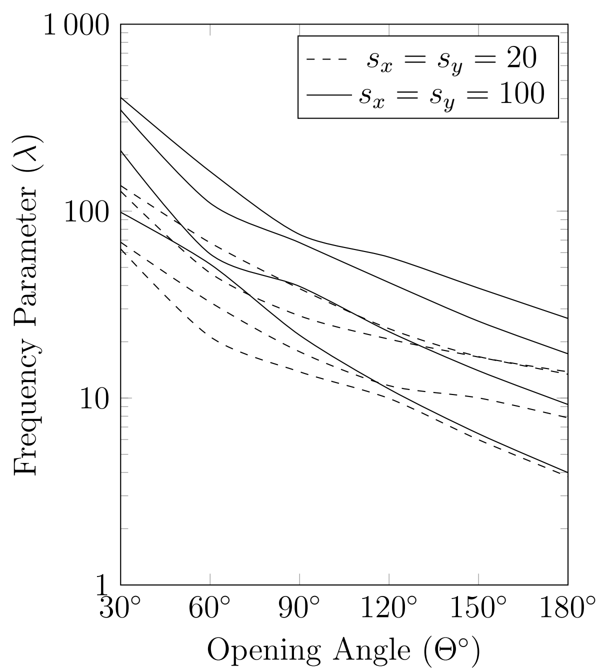

Figure 4.1: Out-of-Plane vibration of F-F Curved Beam

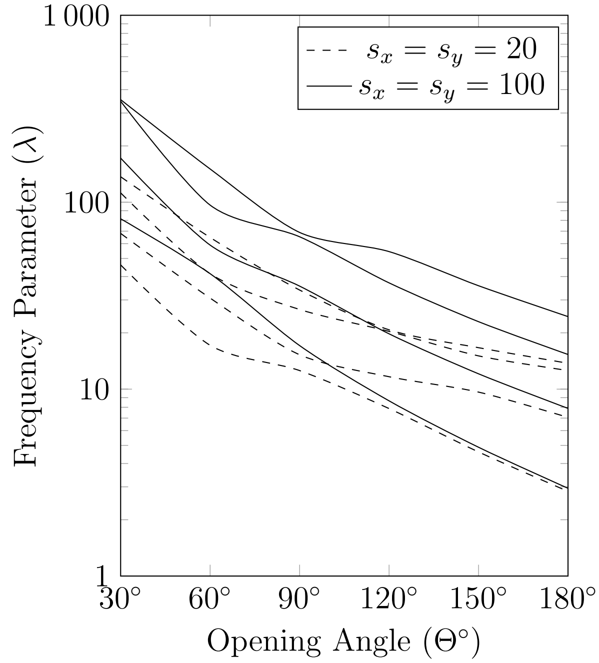

Figure 4.2: Out-of-Plane vibration of F-S Curved Beam

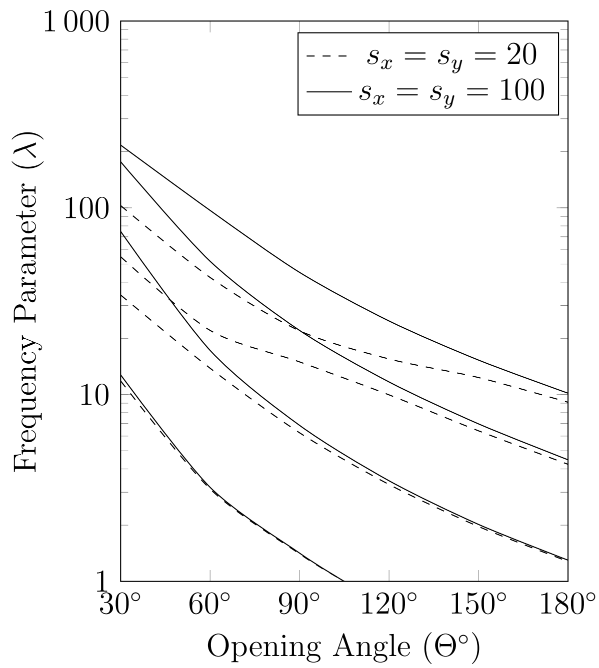

Figure 4.3: Out-of-Plane vibration of F-C Curved Beam

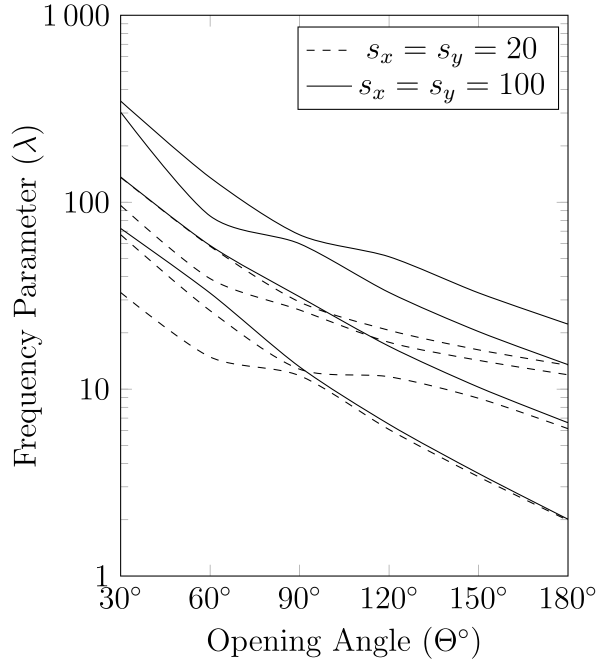

Figure 4.4: Out-of-Plane vibration of S-S Curved Beam

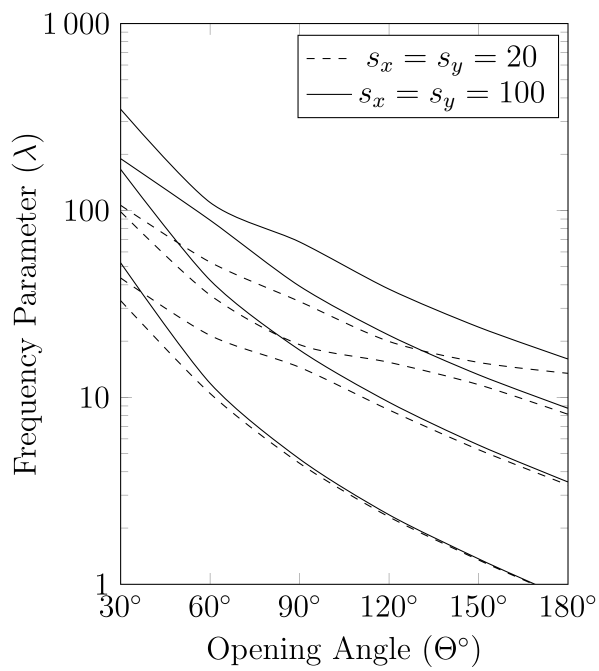

Figure 4.5: Out-of-Plane vibration of S-C Curved Beam

4.1.2 Out-of-plane free vibration analysis of S-shaped curved beam

The out-of-plane natural frequencies of clamped-clamped S-shaped beam (Fig. 4.6) composed of two identical half-rings with circular cross-section are determined by using SEM considering all the effects of shear deformation and rotary inertia. The beam is modeled with the four number of circular beam elements with subtended angle of 90° each.

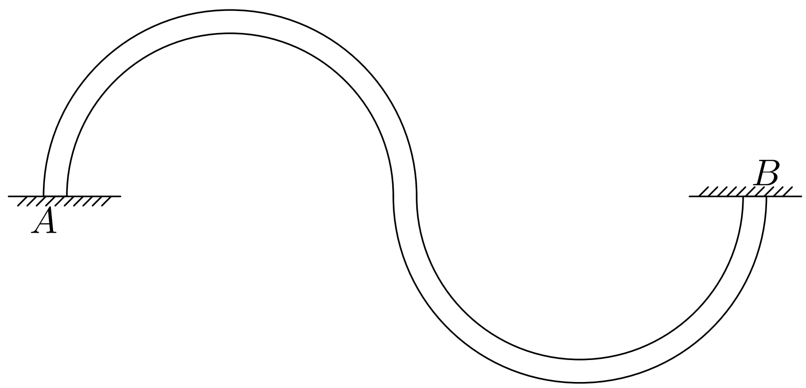

Figure 4.6: C-C S-shaped curved beam

Table 4.4: Out-of-plane frequency parameters of S-shaped curved beam

Slenderness Ratio $s_x = s_y$	Mode Number	Present (SEM) 4 elements	Kim et al. [13] (FEM) 360 elements
20	1	0.59431	0.59431
	2	0.88651	0.88651
	3	2.3955	2.3955
	4	3.4674	3.4674
	5	6.1575	6.1575
	6	8.2507	8.2507

The shear coefficient of circular cross-section κ is 0.89, Poisson's ratio ν is 0.3 and slenderness ratio s_x is 20. The numerical results obtained by 4 number of spectral elements are compared with the 360 number of finite elements by Kim et al. [13] in Table 4.4. It may be observed that these two results are in exact match.

4.1.3 Out-of-plane natural frequencies of continuous curved beam

A two-equal-span semicircular continuous curved beam as shown in Fig. 4.7 is presented to illustrate the application of the Spectral Element Method (SEM). The beam is simply supported over the span AC and continuous at B. The geometrical and physical parameters of the spans AB and BC are identical with $\Theta = \pi/2$, $S_x = 23.39$, $\kappa = 0.83/0.89$ (square / circular cross-section), $\nu = 0.3$ and $\mu = 10^3$. The out-of-plane natural frequency parameters λ for the continuous curved beam with square cross-section and circular cross-section are determined using SEM considering shear deformation and rotary inertia and the results are presented in Table 4.5.

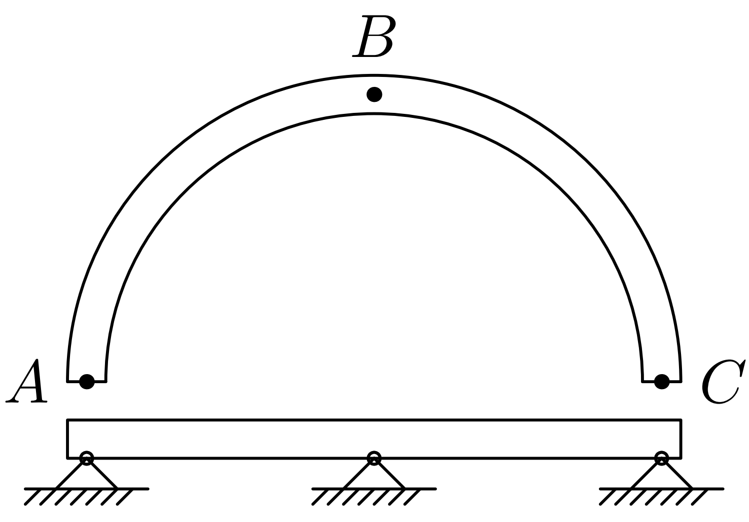

Figure 4.7: A continuous circular curved beam

Table 4.5: Out-of-plane frequency parameters of the continuous circular curved beam

Slenderness Ratio $s_x = s_y$	Mode Number	Square cross-section				Circular cross-section
		Present (SEM)	Exact Solution [10]	DQEM [12]		Present (SEM)
				N=9	N=30	
23.39	1	2.960	2.967	2.965	2.965	2.961
	2	5.384	5.394	5.39	5.39	5.392
	3	14.224	14.24	14.26	14.23	14.259
	4	17.767	17.89	17.91	17.87	17.847
	5	31.255	31.29	31.42	31.26	31.412
	6	35.479	35.57	35.59	35.54	35.736
	7	52.382	52.43	54.12	52.39	52.793
	8	56.550	56.82	60.01	56.77	57.126

For the validation purpose the numerical results of square cross-section are compared with the solutions by exact method of Howson and Jemah [10] and DQEM of Li et al. [12] and are found to be well in agreement. It is found that the natural frequencies of continuous circular beam increases slightly when square cross-section is replaced by circular cross-section of same slenderness ratio.

4.2 Numerical Examples of in-plane vibration of curved beam

4.2.1 Natural frequencies of circular curved beam with different boundary conditions

The natural frequencies of the in-plane vibration of circular curved beam with uniform cross-section under different boundary conditions are computed using SEM considering all the effects of shear deformation and rotary inertia.

Table 4.7 shows the frequency parameters of in-plane vibration of clamped-clamped circular curved beam of square and circular cross sections. The frequencies are for $\Theta = 60°$, $120°$ and $180°$ with slenderness ratios 20 and 100, the shear correction factor $\kappa = 0.85/.89$ (square/ circular cross-section) and Poisson's ratio $\nu = 0.3$. The natural frequencies by SEM are in excellent agreement with those by transfer matrix method (Irie et al. [22]), where it was pointed out that 12.57* must be a typo for 10.57 [9].

In the Table 4.6 in-plane frequency parameters of circular curved beam with circular cross-section under the free-free, free-simple, free-clamped, simple-simple and simple-clamped end restraints are presented for an opening angle of the beam $\Theta = 120°$, shear correction factor $\kappa = 0.89$ for circular cross-section and Poisson's ratio $\nu = 0.3$. The results obtained by SEM are compared with the analytical ones (Irie et al. [22]) which are well in agreement.

Table 4.6: In-plane frequency parameters of circular curved beam with circular cross-section

Slenderness Ratio s_y	Mode Number	Free-Free		Free-Simple		Free-Clamped		Simple-Simple		Simple-Clamped	
		Present (SEM)	Ref. [22]	Present (SEM)	Ref. [22]	Present (SEM)	Ref. [22]	Present (SEM)	Ref. [22]	Present (SEM)	Ref. [22]
10	1	4.136	4.214	2.283	2.250	0.849	-	5.798	5.817	6.940	6.844
	2	10.526	10.671	7.758	7.708	3.142	3.100	8.600	8.603	8.720	8.453
	3	17.555	17.496	12.055	11.915	8.426	8.261	14.599	14.568	15.979	15.689
	4	18.726	18.858	16.470	16.332	12.412	12.413	16.818	16.432	16.833	16.415
100	1	4.505	4.610	2.455	2.425	0.876	-	6.913	6.864	9.178	9.054
	2	12.991	13.060	9.713	9.656	3.658	3.604	17.385	17.191	20.265	20.050
	3	26.237	26.564	21.883	21.401	12.085	12.044	33.509	33.333	37.777	37.065
	4	43.910	44.031	38.455	37.867	25.286	24.843	52.458	52.355	57.023	56.950

Table 4.7: In-plane frequency parameters of clamped-clamped circular curved beam

Cross section	Slenderness Ratio s_y	Mode Number	$\Theta = 60°$ Present (SEM)	$\Theta = 60°$ Reference [22]	$\Theta = 120°$ Present (SEM)	$\Theta = 120°$ Reference [22]	$\Theta = 180°$ Present (SEM)	$\Theta = 180°$ Reference [22]
Circular cross section	20	1	23.757	23.75	10.612	10.61	4.154	4.151
		2	39.023	39.05	15.182	15.19	8.539	8.542
		3	62.635	62.38	24.711	24.72	15.457	15.46
		4	70.698	70.71	30.503	30.47	17.905	17.91
	100	1	52.817	52.82	11.791	11.79	4.375	4.374
		2	76.006	76.01	23.250	23.25	9.603	9.603
		3	117.887	117.9	42.369	42.37	17.809	17.81
		4	171.073	171.1	61.427	61.43	27.216	27.22
Square cross section	20	1	23.709	23.70	10.585	12.57*	4.148	4.143
		2	38.759	38.73	15.174	15.17	8.527	8.519
		3	62.348	62.35	24.648	24.63	15.419	15.40
		4	70.008	69.97	30.391	30.38	17.901	17.90
	100	1	52.786	52.78	11.789	11.79	4.375	4.374
		2	75.979	75.98	23.245	23.24	9.603	9.602
		3	117.812	117.8	42.354	42.35	17.807	17.81
		4	171.812	170.8	61.397	61.39	27.210	27.21

The variation of the natural frequencies of in-plane vibration of circular curved beam with square cross-section with respect to the opening angle Θ for various combination of boundary conditions are shown in Fig. 4.8-4.12 (C-Clamped, F-Free, S-Simple Support). The shear correction factor κ for square cross-section is 0.85 and Poisson's ratio ν is 0.3. The slenderness ratios of the cross-section are taken as 20 and 100.

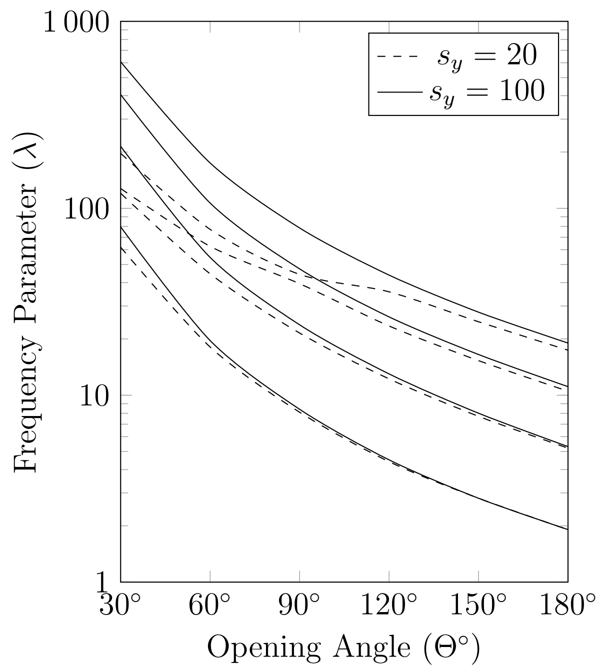

Figure 4.8: In-Plane vibration of F-F Curved Beam

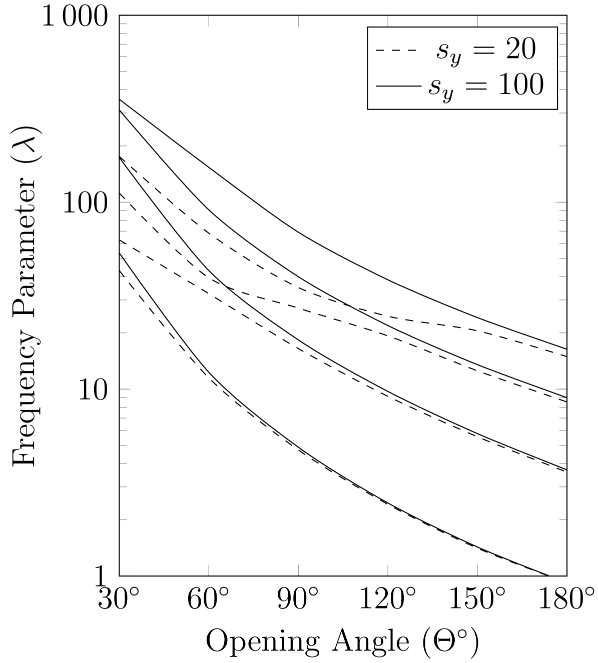

Figure 4.9: In-Plane vibration of F-S Curved Beam

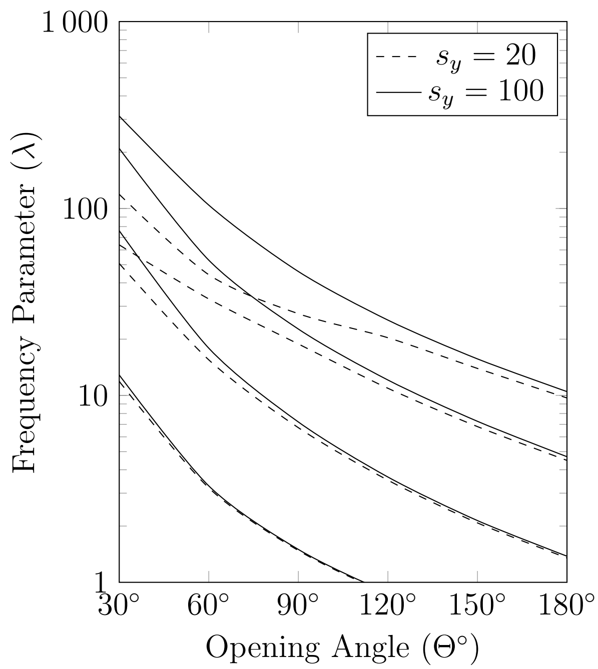

Figure 4.10: In-Plane vibration of F-C Curved Beam

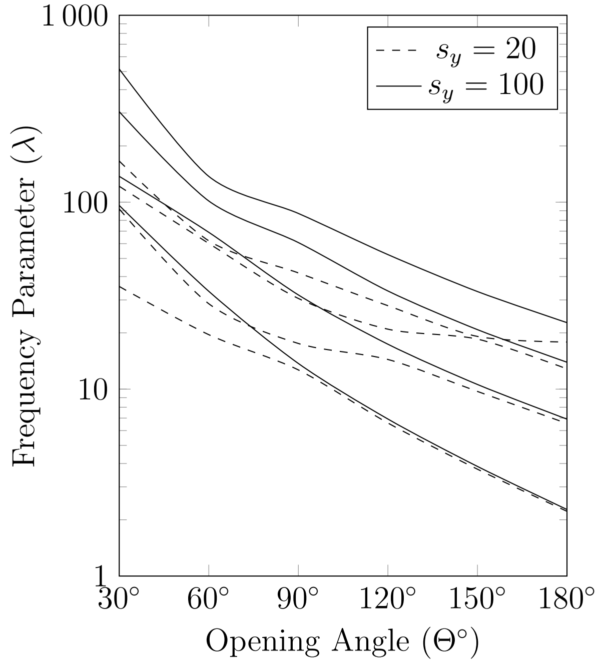

Figure 4.11: In-Plane vibration of S-S Curved Beam

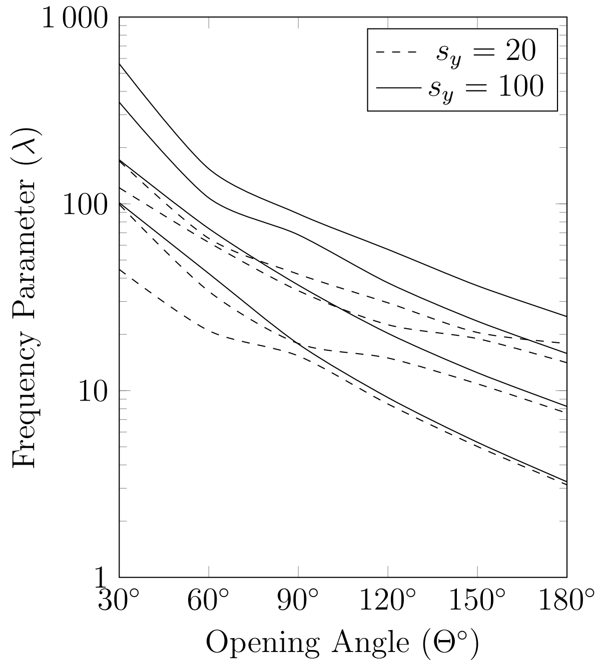

Figure 4.12: In-Plane vibration of S-C Curved Beam

4.2.2 Free in-plane vibration analysis of S-shaped curved beam

The in-plane natural frequencies of clamped-clamped S-shaped curved beam (Fig. 4.6) with circular cross-section composed of two identical half-rings are determined by using SEM considering all the effects of shear deformation and rotary inertia. The beam is modeled with the four numbers of circular beam elements with subtended angle of 90° each. The shear coefficient κ for circular cross-section is 0.89, Poisson's ratio ν is 0.3 and slenderness ratio s_y is 20. The numerical results are obtained for various combination of end restraints and are presented in Table 4.8.

Table 4.8: In-plane frequency parameters of S-shaped curved beam with circular cross-section

Slenderness Ratio s_y	Mode Number	C-C	C-F	C-S	S-S	S-F
20	1	0.6650	0.1170	0.5340	0.3000	0.2810
	2	1.7440	0.5210	1.4160	1.2690	0.6940
	3	3.1300	0.6940	2.7150	2.2130	1.6650
	4	5.7110	2.0750	5.1460	4.6870	3.0750
	5	7.5720	3.4050	7.1160	6.5450	5.3170
	6	11.0370	5.9490	10.4240	9.8980	7.6390
	7	14.1160	8.0690	13.5070	12.8340	10.8760
	8	17.6899	11.5240	17.2280	16.7280	14.0040

(C-Clamped, F-Free, S-Simple Support)

4.2.3 In-plane natural frequencies of continuous curved beam

A two-equal-span semicircular continuous curved beam as shown in Fig. 4.7 is presented to illustrate the application of the Spectral Element Method (SEM). The beam is simply supported over the span AC and continuous at B. The geometrical and physical parameters of the spans AB and BC are identical with $\Theta = \pi/2$, $S_y = 100$, $\kappa = 0.85/0.89$ (square / circular cross-section) and $\nu = 0.3$. The in-plane natural frequency parameters λ for the continuous curved beam with square cross-section and circular cross-section are determined using SEM considering shear deformation and rotary inertia and the results are presented in Table 4.9.

Table 4.9: In-plane frequency parameters of continuous curved beam

Slenderness Ratio s_y	Mode Number	Circular Cross-Section	Square Cross-Section
100	1	13.715	13.715
	2	17.763	17.761
	3	31.919	31.914
	4	36.742	36.732
	5	60.846	60.825
	6	68.300	68.266
	7	87.106	87.081
	8	88.435	88.414

Chapter 5

Conclusion

A circular beam element for both out-of-plane and in-plane free vibration analysis of a circular curved Timoshenko beam by SEM is presented. The spectral elements presented here describe the out-of-plane and in-plane motions of a circular curved beam quite efficiently and accurately in which both the rotary inertia and shear deformation effects are inclusive. In view of the results obtained, one may draw the following conclusions.

1. Both out-of-plane and in-plane natural frequencies by SEM are in excellent agreement with the results by the other methods. Thus SEM is a valid method for free vibration analysis of circular curved beams.

2. This method is easy to implement as it is similar to the conventional Finite Element Method (FEM).

3. The natural frequencies obtained with only a few number of degrees of freedom compare well with the published ones, thus promising high computational efficiency.

4. The frequency parameters become larger with the increase of slenderness ratio and with decrease of the opening angle.

5. The natural frequencies of curved beam with circular cross-section is higher to a small degree than that of square cross-section for the same slenderness ratio.

6. The presented method can be implemented easily and efficiently for precise out-of-plane and in-plane vibration analysis of circular curved Timoshenko beams.

Dissemination of Work

Conferences

1. **Pranab Kumar Ojah**, Manoranjan Barik. *Transverse Free Vibration Analysis of Continuous Horizontally Circular Curved Beam by Spectral Element Method.* International Conference on Applied Engineering, Science & Technology, MIET Madurai, Tamilnadu, India, April 13 2015.

2. **Pranab Kumar Ojah**, Manoranjan Barik. *In-Plane Vibration of Circular Curved Beam by Spectral Element Method.* National Seminar Cum Workshop on Advances in Civil and Infrastructure Engineering (ACIE), Tezpur University, Tezpur, Assam, India May 8-9 2015.

Communicated

Journals

1. **Pranab Kumar Ojah**, Manoranjan Barik. *Transverse Free Vibration of Circular Curved Timoshenko Beam by Spectral Element Method.* Manuscript submitted to **International Journal of Mechanical Sciences**, Elsevier Publication.

2. **Pranab Kumar Ojah**, Manoranjan Barik. *In-Plane Free Vibration of Circular Curved Timoshenko Beam by Spectral Element Method.* Manuscript submitted to **Archives of Civil and Mechanical Engineering**, Elsevier Publication.

Bibliography

[1] GV Narayanan and DE Beskos. Use of dynamic influence coefficients in forced vibration problems with the aid of fast fourier transform. *Computers & Structures*, 9(2):145–150, 1978.

[2] SA Rizzi and JF Doyle. A spectral element approach to wave motion in layered solids. *Journal of Vibration and Acoustics*, 114(4):569–577, 1992.

[3] Augustus Edward Hough Love. *A treatise on the mathematical theory of elasticity.* Cambridge University Press, 1944.

[4] IU Ojalvo. Coupled twist-bending vibrations of incomplete elastic rings. *International Journal of Mechanical Sciences*, 4(1):53–72, 1962.

[5] SS Rao. Effects of transverse shear and rotatory inertia on the coupled twist-bending vibrations of circular rings. *Journal of Sound and Vibration*, 16(4):551–566, 1971.

[6] R Davis, RD Henshell, and GB Warburton. Constant curvature beam finite elements for in-plane vibration. *Journal of Sound and Vibration*, 25(4):561–576, 1972.

[7] T Irie, G Yamada, and K Tanaka. Natural frequencies of out-of-plane vibration of arcs. *Journal of Applied Mechanics*, 49(4):910–913, 1982.

[8] MS Issa. Natural frequencies of continuous curved beams on winkler-type foundation. *Journal of sound and vibration*, 127(2):291–301, 1988.

[9] K Kang, CW Bert, and AG Striz. Vibration analysis of shear deformable circular arches by the differential quadrature method. *Journal of Sound and Vibration*, 183(2):353–360, 1995.

[10] WP Howson and AK Jemah. Exact out-of-plane natural frequencies of curved timoshenko beams. *Journal of Engineering Mechanics*, 125(1):19–25, 1999.

[11] Sen-Yung Lee and Jeng-Chung Chao. Exact solutions for out-of-plane vibration of curved nonuniform beams. *Journal of applied mechanics*, 68(2):186–191, 2001.

[12] LI Hongjing, Tong Wang, and Shuangke Wei. Natural vibration analysis of continuous horizontally curved girder bridges using dqem. *A A*, 1:2–4, 2008.

[13] Bo Yeon Kim, Chang-Boo Kim, Seung Gwan Song, Hyeon Gyu Beom, and Chongdu Cho. A finite thin circular beam element for out-of-plane vibration analysis of curved beams. *Journal of mechanical science and technology*, 23(5):1396–1405, 2009.

[14] S Rajasekaran. Analysis of curved beams using a new differential transformation based curved beam element. *Meccanica*, 49(4):863–886, 2014.

[15] JP Den Hartog. The lowest natural frequency of circular arcs. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 5(28):400–408, 1928.

[16] E Volterra and JD Morell. A note on the lowest natural frequency of elastic arcs. *Journal of Applied Mechanics*, 27(4):744–746, 1960.

[17] Shin Takahashi. Vibration of a circular arc bar in its plane: Both ends built-in. *Bulletin of JSME*, 6(24):666–673, 1963.

[18] AB Sabir and DG Ashwell. A comparison of curved beam finite elements when used in vibration problems. *Journal of Sound and Vibration*, 18(4):555–563, 1971.

[19] DJ Dawe. The transverse vibration of shallow arches using the displacement method. *International Journal of Mechanical Sciences*, 13(8):713–720, 1971.

[20] Shoei-Sheng Chen. In-plane vibration of continuous curved beams. *Nuclear Engineering and Design*, 25(3):413–431, 1973.

[21] SS Rao and V Sundararajan. Erratum:In-plane flexural vibrations of circular rings (Journal of Applied Mechanics, 1969, 36, pp. 620–625). *Journal of Applied Mechanics*, 37(4):1208–1208, 1970.

[22] T Irie, G Yamada, and K Tanaka. Natural frequencies of in-plane vibration of arcs. *Journal of Applied Mechanics*, 50(2):449–452, 1983.

[23] T Irie, G Yamada, et al. In-plane vibration of timoshenko arcs with variable cross-section. *Ingenieur-Archiv*, 48(5):337–346, 1979.

[24] Jinhee Lee. In-plane free vibration analysis of curved timoshenko beams by the pseudospectral method. *KSME international journal*, 17(8):1156–1163, 2003.

[25] Chang-Boo Kim, Park Jung-Woo, Kim Sehee, and Cho Chongdu. A finite thin circular beam element for in-plane vibration analysis of curved beams. *Journal of mechanical science and technology*, 19(12):2187–2196, 2005.

[26] F Yang, R Sedaghati, and E Esmailzadeh. Free in-plane vibration of general curved beams using finite element method. *Journal of sound and vibration*, 318(4):850–867, 2008.

[27] Usik Lee. *Spectral Element Method in Structural Dynamics*. John Wiley & Sons, 2009.

[28] Ivan Stephen Sokolnikoff and Robert Dickerson Specht. *Mathematical theory of elasticity*, volume 83. McGraw-Hill New York, 1956.

[29] GR Cowper. The shear coefficient in Timoshenkos beam theory. *Journal of Applied Mechanics*, 33(2):335–340, 1966.